Eulerian Fluids

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Projection method

Chorin-style projection

$$
\begin{aligned}
\frac{\partial \mathbf{u}}{\partial t} &= -\frac{\Delta t}{\rho}\nabla p \quad \textbf{s.t.} \quad \nabla \cdot \mathbf{u} = \mathbf{0} \\
\mathbf{u}^* - \mathbf{u} &= -\frac{\Delta t}{\rho}\nabla p \quad \textbf{s.t.} \quad \nabla \cdot \mathbf{u}^* = \mathbf{0} \\
\mathbf{u}^* &= \mathbf{u} - \frac{\Delta t}{\rho}\nabla p \quad \textbf{s.t.} \quad \nabla \cdot \mathbf{u}^* = \mathbf{0} \\
\nabla \cdot \mathbf{u}^* &= \nabla \cdot (\mathbf{u} - \frac{\Delta t}{\rho}\nabla p) \\
0 &= \nabla \cdot \mathbf{u} - \frac{\Delta t}{\rho}\nabla^2 p \\
\nabla^2 p &= \frac{\rho}{\Delta t}\nabla \cdot \mathbf{u}
\end{aligned}
$$

On a staggered grid:

$$
\begin{aligned}
\nabla^2 p_{i,j} &= \frac{1}{\Delta x^2}(-4p_{i,j} + p_{i+1,j} + p_{i-1,j} + p_{i,j-1} + p_{i,j+1}) \\
(\frac{\rho}{\Delta t}\nabla \cdot \mathbf{u})_{i,j} &= \frac{\rho}{\Delta t \Delta x}(\mathbf{u}^x_{i+1,j} - \mathbf{u}^x_{i,j} + \mathbf{u}^y_{i,j+1} - \mathbf{u}^y_{i,j})
\end{aligned}
$$

A Fast Variational Framework for Accurate Solid-Fluid Coupling

$$
\begin{gathered}
\dots \\
\frac{\Delta t}{\rho^2}\mathbf{G}^T\mathbf{M}_F\mathbf{G}\mathbf{p} = \frac{1}{\rho}\mathbf{G}^T\mathbf{M}_F\tilde{\mathbf{u}}
\end{gathered}
$$

On a staggered grid (multiply both sides by -1 to make the system positive semi-definite):

$$
\begin{aligned}
-\frac{\Delta t}{\rho^2}\mathbf{G}^T\mathbf{M}_F\mathbf{G}\mathbf{p} &= \frac{\Delta t}{\rho^2\Delta x^2}\Big((m^x_{i+1,j} + m^x_{i,j} + m^y_{i,j+1} + m^y_{i,j})p_{i,j} \\
&\quad - m^x_{i+1,j}p_{i+1,j} - m^x_{i,j}p_{i-1,j} - m^y_{i,j+1}p_{i,j+1} - m^y_{i,j}p_{i,j-1}\Big) \\
&= \frac{\Delta t}{\rho\Delta x^2}\Big((V^x_{i+1,j} + V^x_{i,j} + V^y_{i,j+1} + V^y_{i,j})p_{i,j} \\
&\quad - V^x_{i+1,j}p_{i+1,j} - V^x_{i,j}p_{i-1,j} - V^y_{i,j+1}p_{i,j+1} - V^y_{i,j}p_{i,j-1}\Big) \\
-\frac{1}{\rho}\mathbf{G}^T\mathbf{M}_F\tilde{\mathbf{u}} &= \frac{1}{\rho\Delta x}(-m^x_{i+1,j}\mathbf{u}^x_{i+1,j} + m^x_{i,j}\mathbf{u}^x_{i,j} - m^y_{i,j+1}\mathbf{u}^y_{i,j+1} + m^y_{i,j}\mathbf{u}^y_{i,j}) \\
&= \frac{1}{\Delta x}(-V^x_{i+1,j}\mathbf{u}^x_{i+1,j} + V^x_{i,j}\mathbf{u}^x_{i,j} - V^y_{i,j+1}\mathbf{u}^y_{i,j+1} + V^y_{i,j}\mathbf{u}^y_{i,j})
\end{aligned}
$$

Then, we get $\mathbf{u}^{n+1}$ same like the projection method:

$$
\mathbf{u}^{n+1} = \mathbf{u}^* - \frac{\Delta t}{\rho}\nabla p
$$

After coupling with the rigid:

$$(\frac{\Delta t}{\rho^2}\mathbf{G}^T\mathbf{M}_F\mathbf{G} + \Delta t\mathbf{J}^T\mathbf{M}_S^{-1}\mathbf{J})\mathbf{p} = \frac{1}{\rho}\mathbf{G}^T\mathbf{M}_F\tilde{\mathbf{u}} - \mathbf{J}^T\mathbf{V}^n$$

Why the volume V is in range $[0,1]$? This is correct. We can first use the real V, then we scale all V to range $[0,1]$ by a factor on both sides of the equations.

Derivation of J

let n is the number of cells.

$\mathbf{J}$ is 3*n in 2D, and 6*n in 3D.

$\mathbf{V}^n$ is 3*1 in 2D, and 6*1 in 3D.

$\mathbf{Jp}$ is 3*1 in 2D, and 6*1 in 3D.

$\mathbf{J}^T\mathbf{V}^n$ is n*1 in 2D and 3D.

pressure after integrating is force, this is strictly correct, not a casual idea.

The notation $\mathbf{J}p$ in Eq(2) does not mean the applied force from fluid to solid in every cell is $\mathbf{J}$ multiply the pressure **in this cell**, it actually means "converting **pressure on the boundary** of the solid to force". This is a global concept, it is not applicable in an individual cell. **The actual force from fluid to solid in a cell is not a mentioned concept in this paper.** All calculation of forces from fluid to solid is described in a global view. Since the force applied to solid is an integration of pressure, there is no concept of **"force in a cell"**. Thus after we get $\mathbf{J}$, we can not regard $\mathbf{J}p_{i,j}$ as the force in a single cell, it is meaningless. In conclusion, $\mathbf{J}$ is meaningless.

We can getting every term of $\mathbf{J}$ from Eq(10):

$$\begin{aligned}
J_x\mathbf{p} &= -\sum_{i,j} V^x_{i,j}\frac{p_{i+1,j}-p_{i,j}}{\Delta x} \\
&= V^x_{0,0}\frac{p_{1,0}-p_{0,0}}{\Delta x} + V^x_{1,0}\frac{p_{2,0}-p_{1,0}}{\Delta x} + \cdots \\
&= -\frac{V^x_{0,0}p_{0,0}}{\Delta x} + \frac{V^x_{0,0}p_{1,0} - V^x_{1,0}p_{1,0}}{\Delta x} + \cdots
\end{aligned}$$

Unify the first and the last term (this is indeed correct in the implementation since the first and the last cell are usually not involved in the calculation because they are usually not water), we have

$$\begin{aligned}
J_x\mathbf{p} &= -\sum_{i,j}\frac{V^x_{i,j}-V^x_{i+1,j}}{\Delta x}p_{i,j} \\
&= \sum_{i,j}\frac{V^x_{i+1,j}-V^x_{i,j}}{\Delta x}p_{i,j}
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
J_{rot}\mathbf{p} &= \sum_{i,j}\left[\left(V^x_{i,j}\frac{p_{i+1,j}y_{i+1,j}-p_{i,j}y_{i,j}}{\Delta x}\right) - \left(V^y_{i,j}\frac{p_{i,j+1}x_{i,j+1}-p_{i,j}x_{i,j}}{\Delta x}\right)\right] \\
&= \sum_{i,j}\left(\frac{V^y_{i,j+1}-V^y_{i,j}}{\Delta x}p_{i,j}x_{i,j} - \frac{V^x_{i+1,j}-V^x_{i,j}}{\Delta x}p_{i,j}y_{i,j}\right)
\end{aligned}$$

Thus

$$\mathbf{J} = \begin{bmatrix}\mathbf{J}_x \\ \mathbf{J}_y \\ \mathbf{J}_{rot}\end{bmatrix} = \begin{bmatrix}\frac{V^x_{i+1,j}-V^x_{i,j}}{\Delta x} \\ \frac{V^y_{i,j+1}-V^y_{i,j}}{\Delta x} \\ \mathbf{J}_y x_{i,j} - \mathbf{J}_x y_{i,j}\end{bmatrix} \tag{1}$$