Derivation of Navier-Stokes Equations

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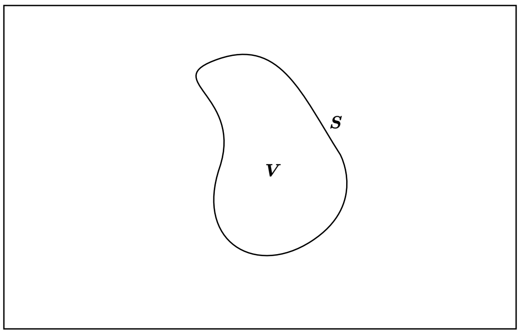
We drive the Euler equations - simplified NS equations without viscosity term. The derivation is based on mass and (linear) momentum (and energy) conservation. Our derivation will be from an Eulerian viewpoint.



Mass Conservation

$\underline{\textit{The change rate of mass}} + \underline{\textit{The flux of mass}} = 0$

The change rate of mass:

$$\frac{\partial m}{\partial t} = \frac{\partial(\int_V \rho dV)}{\partial t} = \int_V \frac{\partial \rho}{\partial t} dV \tag{1}$$

The flux of mass:

$$\int_S \rho \mathbf{u} \cdot \mathbf{n} dS = \int_V \nabla \cdot (\rho \mathbf{u}) dV \tag{2}$$

From equations 1 and 2, we have

$$\int_V \frac{\partial \rho}{\partial t} dV + \int_V \nabla \cdot (\rho \mathbf{u}) dV = 0$$

$\Longrightarrow$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \tag{3}$$

Momentum Conservation

$\underline{\textit{The change rate of momentum}} + \underline{\textit{The flux of momentum}} = \underline{\textit{Internal impulse}}$ - $\underline{\textit{External impulse}}$.

$$\frac{\partial (m\mathbf{v})}{\partial t} + \int_S (\rho \mathbf{v} \cdot \mathbf{n} dS)\mathbf{v} = \int_V \rho \mathbf{f} dV - \int_S p \mathbf{n} dS$$

$\Longrightarrow$

$$\int_V \frac{\partial (\rho \mathbf{v})}{\partial t} dV + \int_V \nabla \cdot (\rho \mathbf{v} \cdot \mathbf{n} dV)\mathbf{v} = \int_V \rho \mathbf{f} dV - \int_V \nabla p dV$$

$\Longrightarrow$

$$\frac{\partial (\rho \mathbf{v})}{\partial t} + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v}) = \rho \mathbf{f} - \nabla p \tag{4}$$

Incompressible Fluids

For Incompressible fluids, ρ is a constant. Thus, equations 3 and 4 can be written as

$$\begin{cases} \nabla \cdot \mathbf{u} = 0 \\ \frac{\partial \mathbf{v}}{\partial t} + \nabla \cdot (\mathbf{v} \otimes \mathbf{v}) = \mathbf{f} - \frac{1}{\rho} \nabla p \end{cases}$$

Energy Conservation

Compressible Fluids