A Derivation of Stable Neo-Hookean Constitutive Model

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References

Original paper: Stable Neo-Hookean Flesh Simulation
Siggraph course: Dynamic Deformables: Implementation and Production Practicalities

Symbols

$$I_2 = tr(\mathbf{F}^T\mathbf{F})$$
$$I_3 = det(\mathbf{F})$$

A Corrected Derivation

We start with the energy density equation:

$$\Psi_D = \frac{\mu}{2}(I_2 - 3) + \frac{\lambda}{2}(I_3 - 1)^2$$

In the siggraph course (Section 6.3.3, page 85), $\frac{\partial \Psi_D}{\partial \mathbf{F}}$ is given as the following form:

$$\frac{\partial \Psi_D}{\partial \mathbf{F}} = \mu\mathbf{F} - \lambda(I_3 - 1)\frac{\partial I_3}{\partial \mathbf{F}}$$

However, the above equation is wrong. The corrected version is:

$$\frac{\partial \Psi_D}{\partial \mathbf{F}} = \mu\mathbf{F} + \lambda(I_3 - 1)\frac{\partial I_3}{\partial \mathbf{F}}$$

In this case, we can re-derive all other equations. A corrected derivation is as follows:

$$\frac{\partial \Psi_\alpha}{\partial \mathbf{F}} = \mu\mathbf{F} + \lambda(I_3 - \alpha)\frac{\partial I_3}{\partial \mathbf{F}}$$

Choose $\mathbf{F} = \mathbf{I}$, we have

$$\begin{aligned}\frac{\partial \Psi_\alpha(\mathbf{I})}{\partial \mathbf{F}} &= \mu\mathbf{I} + \lambda(I_3 - \alpha)\frac{\partial I_3}{\partial \mathbf{I}} \\ &= \mu\mathbf{I} + \lambda(1 - \alpha)\mathbf{I}\end{aligned}$$

To satisfy "rest stability", we let $\frac{\partial \Psi_\alpha(\mathbf{I})}{\partial \mathbf{F}} = 0$, then

$$0 = \mu + \lambda(1 - \alpha)$$

Thus, we have $\alpha = 1 + \frac{\mu}{\lambda}$.
The final energy density function is still identical to the original form given in the course:

$$\boxed{\Psi_{SNH} = \frac{\mu}{2}(I_2 - 3) + \frac{\lambda}{2}(I_3 - 1)^2 - \mu(I_3 - 1)}$$

The corrected PK1 tensor is

$$\boxed{\frac{\partial \Psi_{SNH}}{\partial \mathbf{F}} = \mu\mathbf{F} + \lambda(I_3 - \alpha)\frac{\partial I_3}{\partial \mathbf{F}} - \mu\frac{\partial I_3}{\partial \mathbf{F}}}$$